Endogenous growth

• Properties of the aggregate production $Y = F(K, L, A)$ ← technology.

Neoclassical growth models:

→ Constant returns to scale
- "REPLICATION ARGUMENT"
- FIRMS OPTIMIZE AND THUS SHOULD BE EXPECTED TO OPERATE AT OPTIMAL SCALE
- EMPIRICAL STUDIES TYPICALLY DON'T REJECT CRS

→ decreasing returns to reproducible inputs (e.g., capital)

Def. We say that a model features endogenous growth if it possesses a solution along which all key economic variables grow perpetually, and the long-run growth rate is pinned down by variables determined within the model.

TYPES / CLASSES OF ENDOGENOUS GROWTH MODELS

1. models where long-run growth is driven by accumulation of reproducible inputs only
   - → AK model, Jones-Manuelli model [K]
   - → Uzawa-Lucas model [H]
   - → models where growth is driven by (appropriately specified) externalities
   - → models with multiple reproducible inputs
2. models with endogenous technological change
   - → to which we return later.

Let's assume that the aggregate production function has two inputs only (K, L), constant returns to scale, and constant technology.

$$Y = F(K,L) = L\,F(\underbrace{\tfrac{K}{L}}_{k}, 1) = L\,f(k)$$

$$y = \frac{Y}{L} = f(k).$$

Assume that:

s — "endogenous" variable

1° $f(0) = 0$

2° $f'(k) > 0$

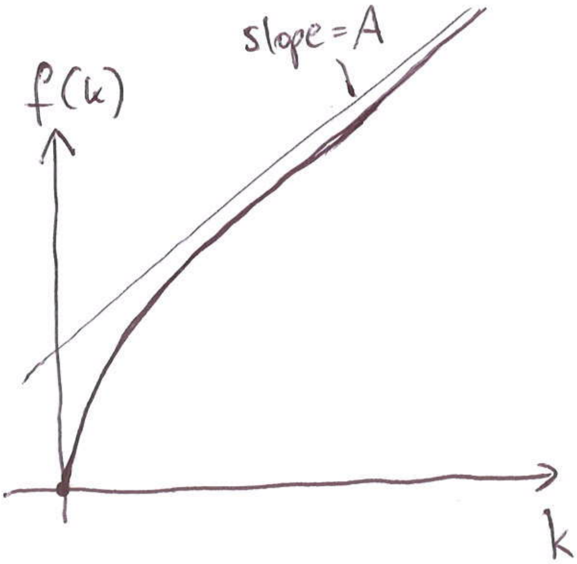
3° $f''(k) \leq 0$

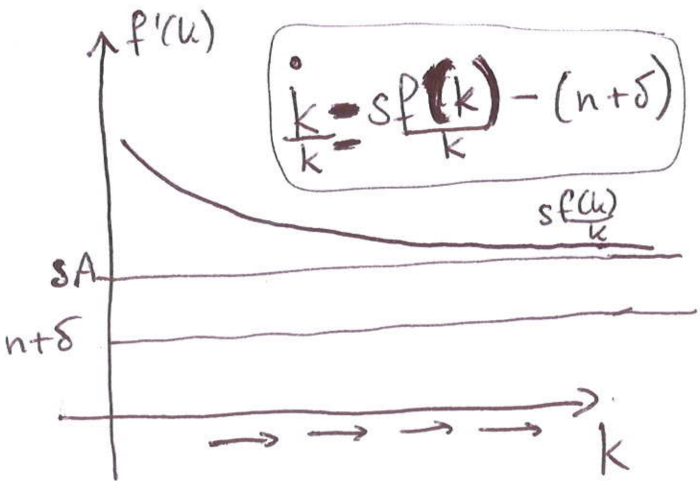
≈ neoclassical assumptions

but relax $\lim_{k\to\infty} f'(k) = 0$

[Inada condition]

Assume instead $f'(k) \xrightarrow[k\to\infty]{} A > 0$.





Examples of relevant production functions:

(1) AK function: $f(k) = Ak$, $F(K,L) = AK$

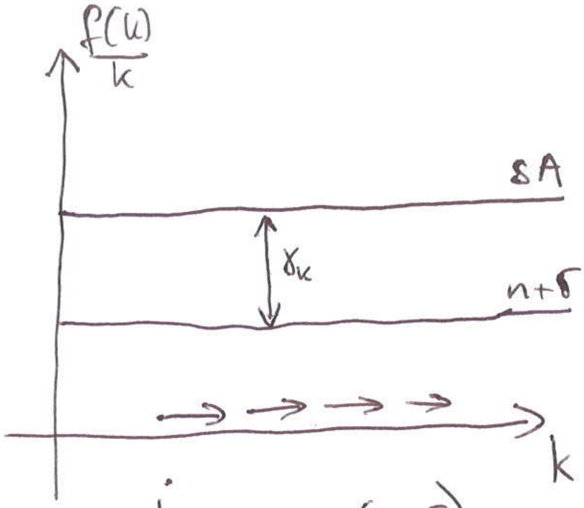
(2) Jones-Manuelli function: $f(k) = Ak + Bk^{\alpha}$, $F(K,L) = AK + BK^{\alpha}L^{1-\alpha}$

(3) CES production function with $\sigma > 1$: $f(k) = A\left[\pi k^{\frac{\sigma-1}{\sigma}} + (1-\pi)\right]^{\frac{\sigma}{\sigma-1}}$,

$$F(K,L) = A\left[\pi K^{\frac{\sigma-1}{\sigma}} + (1-\pi) L^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

Ad 1

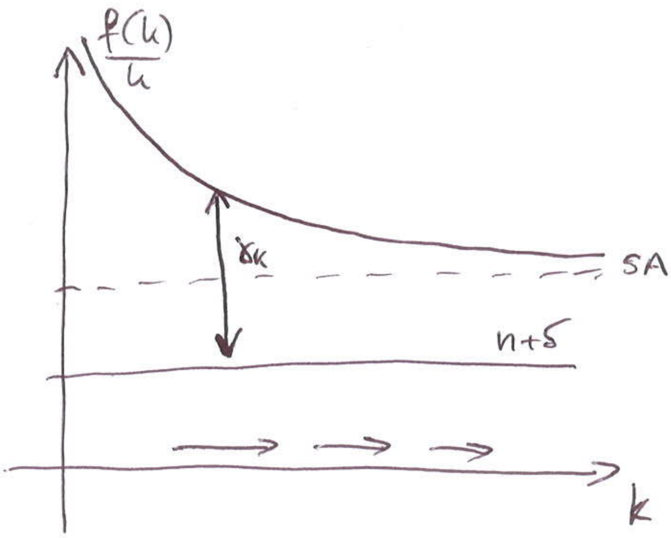
$$f(k) = Ak$$
$$f'(k) = A$$
$$f''(k) = 0$$
$$\lim_{k \to \infty} f'(k) = A$$
$$s\frac{f(k)}{k} = sA.$$



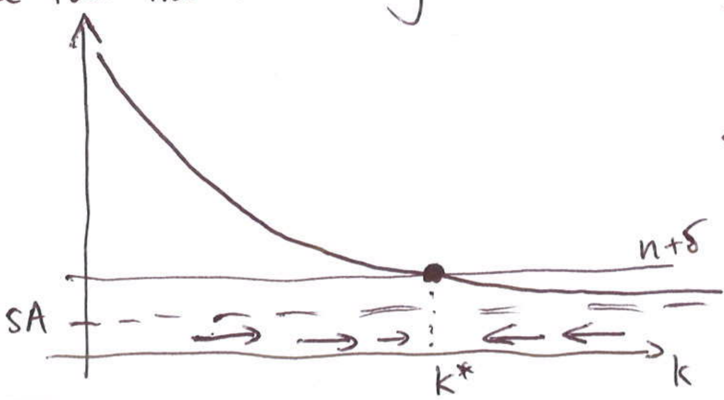
- The growth rate is: $\gamma_k = \frac{\dot{k}}{k} = sA - (n+\delta)$.
- There are no transitional dynamics.
- We assume that A is "sufficiently large".

Ad 2 $\quad f(k) = Ak + Bk^{\alpha}$ (Jones & Manuelli, 1990)

$$f'(k) = A + B\alpha k^{\alpha-1}$$
$$f''(k) = B\alpha \underbrace{(\alpha-1)}_{-} k^{\alpha-2} < 0$$
$$\lim_{k \to \infty} f'(k) = A$$
$$s\frac{f(k)}{k} = sA + \frac{sBk^{\alpha}}{k}.$$



- the growth rate is $\gamma_k = \frac{\dot{k}}{k} = sA + sBk^{\alpha-1} - (n+\delta)$
- Transitional dynamics: γ_k decreases with k (in time).
- A has to be sufficiently large for endogenous growth. Otherwise the model converges to a steady state.



← like in the Solow model!

Ad 3 $f(k) = A\left[\pi k^{\frac{\sigma-1}{\sigma}} + (1-\pi)\right]^{\frac{\sigma}{\sigma-1}}$

(Constant elasticity of substitution $= \sigma$)

$\sigma > 1 \iff$ K and L are gross substitutes

$\sigma < 1 \iff$ K and L are gross complements

$\sigma = 1 \iff$ Cobb-Douglas case

$$f'(k) = \cancel{\frac{\sigma}{\sigma-1}} A\left[\cdot\right]^{\frac{\sigma}{\sigma-1}-1} \cdot \pi \cancel{\left(\frac{\sigma-1}{\sigma}\right)} k^{\frac{\sigma-1}{\sigma}-1} > 0$$

$$f''(k) = A\Bigg[\overbrace{\left(\frac{\sigma}{\sigma-1}-1\right)}^{1/\sigma-1}\left[\cdot\right]^{\frac{\sigma}{\sigma-1}-2} \cdot \pi\left(\frac{\sigma-1}{\sigma}\right)k^{\frac{\sigma-1}{\sigma}-1} \cdot \pi k^{\frac{\sigma-1}{\sigma}-1} +$$

$$+ \left[\cdot\right]^{\frac{\sigma}{\sigma-1}-1}\pi \overbrace{\left(\frac{\sigma-1}{\sigma}-1\right)}^{-1/\sigma} k^{\frac{\sigma-1}{\sigma}-2}\Bigg] =$$

$$= A\pi\left[\cdot\right]^{\frac{\sigma}{\sigma-1}-1} k^{\frac{\sigma-1}{\sigma}-2}\left[\frac{1}{\sigma}\,\frac{\pi k^{\frac{\sigma-1}{\sigma}}}{\pi k^{\frac{\sigma-1}{\sigma}}+(1-\pi)} - \frac{1}{\sigma}\right] < 0$$

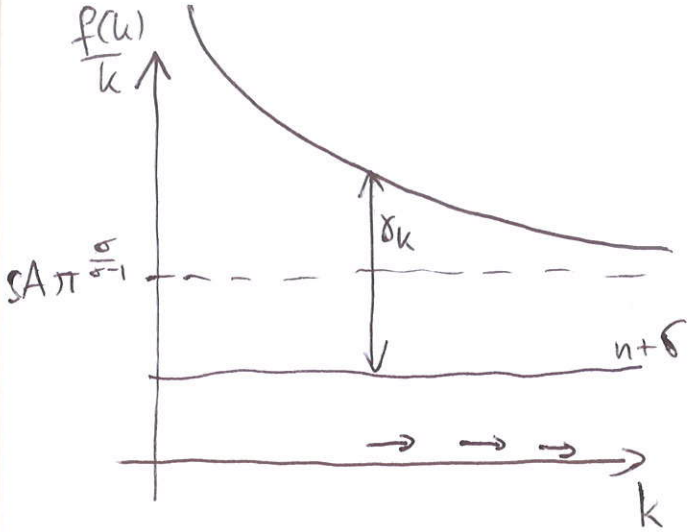
$$\lim_{k\to\infty} f'(k) = \lim_{k\to\infty} A\pi\left[\cdot\right]^{\frac{1}{\sigma-1}} k^{-\frac{1}{\sigma}} =$$

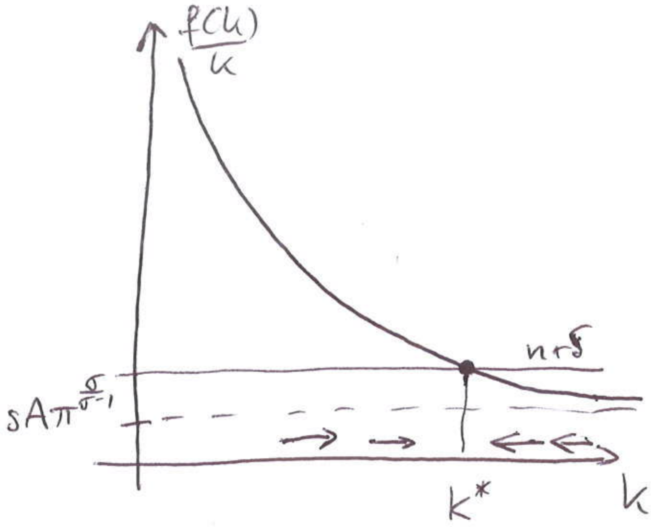
$$= \lim_{k\to\infty} A\pi\left[\pi k^{\frac{\sigma-1}{\sigma}} + (1-\pi)\right]^{\frac{1}{\sigma-1}} \cdot k^{-\frac{1}{\sigma}\left(\frac{\sigma-1}{\sigma-1}\right)} =$$

$$= \lim_{k\to\infty} A\pi\left[\pi + (1-\pi)k^{-\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}} = \begin{cases} A\pi^{\frac{\sigma}{\sigma-1}}, & \sigma > 1 \\ 0, & \sigma < 1 \end{cases}$$

$\boxed{\sigma > 1}$

- The growth rate is $\gamma_k = \frac{\dot{k}}{k} = sA\left[\pi + (1-\pi)k^{-\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} - (n+\delta)$
- There are transitional dynamics, γ_k converges to
  $\lim_{k\to\infty} \gamma_k = sA\pi^{\frac{\sigma}{\sigma-1}} - (n+\delta)$
- We assume that A is "sufficiently large". Otherwise the model converges to a steady state.



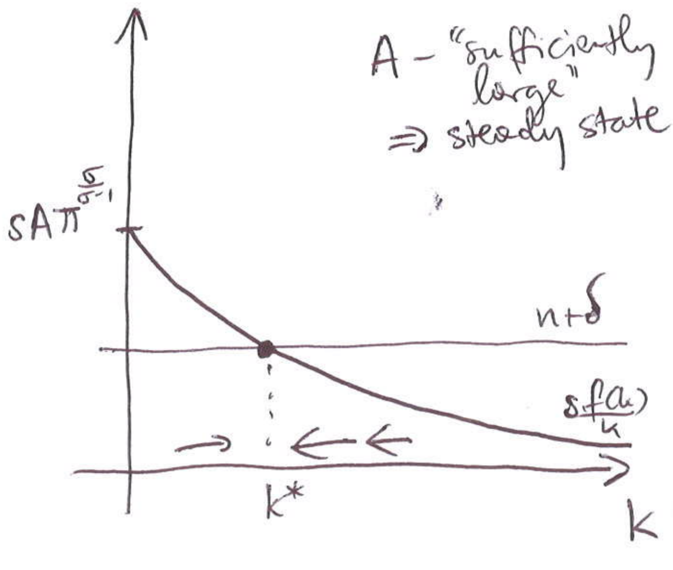


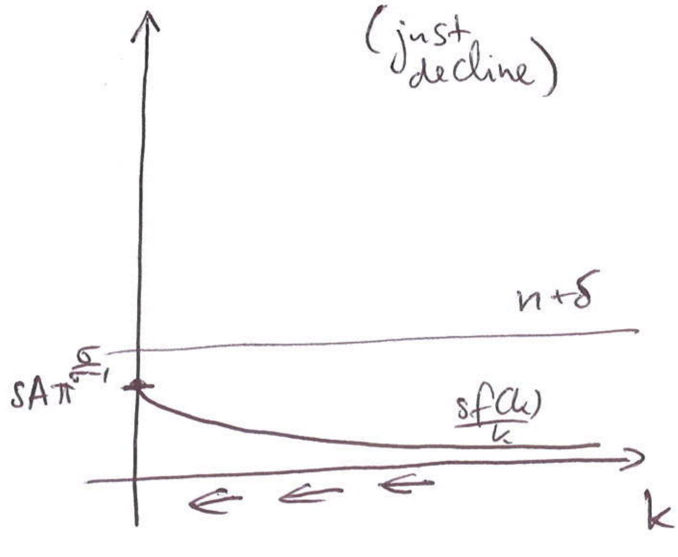
$\boxed{\sigma<1}$

Note that $\lim_{k\to\infty} f'(k) = 0$

and $\lim_{k\to 0} s\frac{f(k)}{k} = \lim_{k\to 0} sA\left[\pi + (1-\pi)\, k^{-\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} = sA\pi^{\frac{\sigma}{\sigma-1}}$

[Inada condition at 0 doesn't hold.]





- In any case, $\sigma<1$ precludes endogenous growth.
  - ↳ AS LONG THERE IS NO OTHER SOURCES OF GROWTH, E.G. TECHNOLOGICAL PROGRESS.

The AK endogenous growth model

- KEY MISSING ELEMENT (so far): endogenous savings rate $\underline{s}$

- <u>Households</u>

$$\max \int_0^\infty e^{-(\rho-n)t} u(c)\,dt \,, \quad \text{where } u(c) = \frac{c^{1-\theta}-1}{1-\theta} \quad (\text{CRRA})$$

subject to $\dot{a} = (r-n)a + w - c$.

We set up the Hamiltonian:

$$\mathcal{H} = \frac{c^{1-\theta}-1}{1-\theta} \cdot e^{-(\rho-n)t} + \lambda\Big((r-n)a + w - c\Big)$$

$$\frac{\partial \mathcal{H}}{\partial c} = c^{-\theta} e^{-(\rho-n)t} - \lambda = 0 \implies \lambda = c^{-\theta} e^{-(\rho-n)t}$$

$$\frac{\partial \mathcal{H}}{\partial a} = \lambda(r-n) = -\dot{\lambda} \implies \hat{\lambda} = \frac{\dot{\lambda}}{\lambda} = -(r-n)$$

$$\Downarrow$$

$$\hat{\lambda} = -\theta\hat{c} - (\rho-n) \implies +\theta\hat{c} + (\rho - \cancel{n}) = r - \cancel{n}$$

$$\boxed{\hat{c} = \frac{r-\rho}{\theta}}$$

Euler equation.

We also require the transversality condition:

$$\lim_{a\to\infty} \lambda a = \lim_{a\to\infty} a(t)\cdot e^{-\int_0^t (r(v)-n)\,dv} = 0.$$

Firms (perfect competition)

$$\max_{K,L} \{ F(K,L) - \tilde{r}K - wL \}$$

implies $\tilde{r} = \frac{\partial F}{\partial K}$, $w = \frac{\partial F}{\partial L}$

Here, $F(K,L) = AK$, so $\underbrace{\tilde{r} = A}_{\text{gross rental price of } K}$, $w = 0$ (no labor in production).

Note that profits are zero.

Equilibrium

- Capital market clears, $a = k$.

$\hookrightarrow \dot{k} = y - c - (\delta + n)k = (A - \delta - n)k - c$

$\hookrightarrow \dot{a} = (r - n)a + w - c = (r - n)k - c$

$$\left.\right\} \quad r = A - \delta, \quad \underbrace{\tilde{r}}_{\text{GROSS}} = \underbrace{r}_{\text{NET}} + \delta$$

RENTAL PRICE

Hence, $\underbrace{\hat{c} = \frac{\dot{c}}{c} = \frac{A - \delta - \rho}{\theta}}$. $\boxed{\text{GROWTH RATE OF THE ECONOMY} \;\; !!!}$

- Observe that the growth rate does <u>not</u> depend on k, and is fixed throughout (no transitional dynamics).
- Solving, we obtain a closed-form solution for $c(t)$:

$$c(t) = c(0)\, e^{\left(\frac{A-\delta-\rho}{\theta}\right)t}.$$

- It is also easy to compute:

$$\underbrace{\frac{c}{k}}_{\text{Constant}} = (A - \delta - n) - \underbrace{\frac{\dot{k}}{k}}_{\text{Constant}} = (A - \delta - n) - \frac{\dot{c}}{c} = \underbrace{\frac{A-\delta}{\theta}(\theta - 1) + \frac{\rho}{\theta} - n}_{\varphi > 0 \ [\text{Ass.}]}.$$

- And so $c(0) = \varphi k(0)$, where <u>$k(0)$ is given</u>.

The savings rate is

$$s = \frac{y-c}{y} = \frac{Ak-c}{Ak} = \frac{Ak-\varphi k}{Ak} = \frac{A-\varphi}{A} =$$

$$= \frac{A-\rho+\theta n+(\theta-1)\delta}{\theta A}.$$

The transversality condition: $\left[g = \frac{A-\delta-\rho}{\theta}\right]$

$$\lim_{k\to\infty} k(0)e^{gt}e^{-\int_0^t (A-\delta-n)dv} = k(0)\,e^{(g-A+\delta+n)t} =$$

$$= k(0)\,e^{\left(\frac{A-\delta}{\theta}(1-\theta)-\frac{\rho}{\theta}+n\right)t} = k(0)e^{-\varphi t} = 0$$

because we have assumed that $\varphi > 0$.

$\square$

Jones & Manuelli (1990) model

- THE HOUSEHOLD'S PROBLEM IS EXACTLY THE SAME

Firms (perfect competition)

$$\tilde{r} = \frac{\partial F}{\partial K}, \quad w = \frac{\partial F}{\partial L}.$$

Here, $F(K,L) = AK + BK^{\alpha}L^{1-\alpha}$, so

$$\tilde{r} = A + B\alpha K^{\alpha-1}L^{1-\alpha}$$
$$w = B(1-\alpha)K^{\alpha}L^{-\alpha}.$$

$$\left[\begin{array}{l} y = f(k) = Ak + Bk^{\alpha} \\ \tilde{r} = A + B\alpha k^{\alpha-1} \\ w = B(1-\alpha)k^{\alpha} \end{array}\right\} \begin{array}{l} y = \tilde{r}k + w, \\ Y = \tilde{r}K + wL. \end{array}$$

Zero profits.

Equilibrium

- Capital market clears, $a = k$.

$$r = \tilde{r} - \delta = A + B\alpha k^{\alpha-1} - \delta$$

- Hence,

$$\underbrace{\hat{c} = \frac{\dot{c}}{c} = \frac{r-\rho}{\theta} = \frac{A + B\alpha k^{\alpha-1} - \delta - \rho}{\theta}}$$

GROWTH RATE OF THE ECONOMY

!!!

Note that $\hat{c}$ depends on k, and

$$\lim_{k\to\infty} \hat{c} = \frac{A-\delta-\rho}{\theta}$$

(as in the AK model).

We have

$$\frac{\dot{k}}{k} = \frac{y}{k} - \frac{c}{k} - (n+\delta) = A + Bk^{\alpha-1} - \frac{c}{k} - (n+\delta)$$

If $k \to \infty$ then $Bk^{\alpha-1} \to 0$ and so $\frac{\dot{k}}{k} \xrightarrow[k\to\infty]{} A - \frac{c}{k} - (n+\delta)$,

and so endogenous growth implies asymptotical constancy of $\frac{c}{k}$,

and thus as $k \to \infty$, $\hat{c} = \hat{k} = \hat{y} = \frac{A-\delta-\rho}{\theta}$.

It follows that $\frac{c}{k} \xrightarrow[k\to\infty]{} A - n - \delta - \frac{A-\delta-\rho}{\theta} = \underbrace{\frac{A-\delta}{\theta}(\theta-1) + \frac{\rho}{\theta} - n}_{\varphi > 0 \ [\text{Ass.}]}$.

Also $\frac{y}{k} \xrightarrow[k\to\infty]{} A$.

However, there are transitional dynamics.

Consider the system:

$$\begin{cases} \dot{k} = Ak + Bk^{\alpha} - c - (n+\delta) \\ \dot{c} = \frac{1}{\theta}(A + B\alpha k^{\alpha-1} - \delta - \rho) \end{cases}$$

We have shown that this system doesn't possess a steady state ($\dot{k} = \dot{c} = 0$).

Let's rewrite it in "stationary" variables — ones that do possess a steady state.

For example, $\begin{cases} u = \frac{c}{k} & [\text{control-like variable}] \\ z = \frac{y}{k} & [\text{state-like variable}]. \end{cases}$

(there exists a given $z(0)$ but not $u(0)$)

$$\begin{cases} \hat{u} = \hat{c} - \hat{k} \\ \hat{z} = \hat{y} - \hat{k} = \cancel{A + Bk^{\alpha-1} - \hat{k}} \\ \quad = \widehat{\left(A + Bk^{\alpha-1}\right)} = \dfrac{\overset{\frac{1}{k}(\alpha-1)(z-A)}{\overbrace{B(\alpha-1)k^{\alpha-2}}}\,\dot{k}}{\underbrace{A + Bk^{\alpha-1}}_{z}} = \hat{k}(\alpha-1)\dfrac{z-A}{z} \end{cases}$$

(Let us also note that

$$A + B\alpha k^{\alpha-1} = \underbrace{\alpha A + B\alpha k^{\alpha-1}}_{\alpha z} + A - \alpha A = \alpha z + (1-\alpha)A.)$$

$$\begin{cases} \hat{u} = \dfrac{1}{\theta}\left(\alpha z + (1-\alpha)A - \delta - \rho\right) - z + u + (n+\delta) = \\ \quad = u + z\left(\dfrac{\alpha}{\theta} - 1\right) + \dfrac{1-\alpha}{\theta}A - \dfrac{\delta+\rho}{\theta} + n + \delta \\ \hat{z} = \left(z - u - (n+\delta)\right)(\alpha-1)\dfrac{z-A}{z} \end{cases}$$

The steady state in the (u,z) space:

$$\hat{u} = \hat{z} = 0 \iff \begin{cases} u + z\left(\dfrac{\alpha}{\theta} - 1\right) = -\left(\dfrac{1-\alpha}{\theta}A - \dfrac{\delta+\rho}{\theta} + n + \delta\right) \\ (z-A)\left(z - u - (n+\delta)\right) = 0 \end{cases}$$

$$\Updownarrow$$

$$\underbrace{z = A}_{(*)} \quad \text{or} \quad \underbrace{z = u + (n+\delta)}_{(**)}$$

Case $(*)$: $\begin{cases} z = A \\ u + A\left(\dfrac{\alpha}{\theta} - 1\right) = -\left(\dfrac{1-\alpha}{\theta}A - \dfrac{\delta+\rho}{\theta} + n + \delta\right) \end{cases}$

$$\begin{cases} z = A \\ u = A\left(1 - \dfrac{1}{\theta}\right) + \dfrac{\delta+\rho}{\theta} - n - \delta \end{cases}$$

AS DISCUSSED EARLIER!!

Case $(**)$: $\begin{cases} z = u + (n+\delta) \\ \dfrac{\alpha}{\theta}\left(u + (n+\delta)\right) = -\dfrac{1-\alpha}{\theta}A - \dfrac{\delta+\rho}{\theta} \end{cases} \Rightarrow u^* < 0$ $\boxed{!}$ CONTRADICTION.